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Linear Approximation and Generalized Convexity

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I. INTRODUCTION

It is well known that the Bernstein polynomials $B_n(f;x)$, $n = 1, 2, \ldots$, of a convex function $f(x)$, satisfy $B_n(f;x) \geqslant f(x)$ for all n and all $x \in [0,1]$ (see e.g. [*11*]). It has also been shown (by B. Averbach) that if $f(x)$ is convex, then the sequence $\{B_n(f;x)\}_{n=0}^{\infty}$ is a monotone decreasing sequence for each fixed x, $x \in [0,1]$ (see [*11*]). These results have been extended to include a class of approximation formulas, by Karlin ([*3*] and [*4*]).

Conversely, it has been stated that if $f(x)$ is continuous on $[0,1]$ and the inequality

$$B_n(f;x) \geqslant f(x), \qquad \text{for all } x \in [0,1],$$

persists for all $n \geqslant 1$, then $f(x)$ is convex (see [*8*]). Furthermore, it has been shown that the condition

$$B_{n+1}(f;x) \leqslant B_n(f;x) \qquad \text{for all } x \in [0,1],\ n = 1,2,\ldots,$$

suffices to ensure the convexity of a twice continuously differentiable function (see [7]).

The main contribution of the present paper is the extension of the converse theorems to quite a wide class of positive linear operators. This is done in Section III.

In Section II we present extensions of the first "direct" theorem, describing the implications of the assumption of convexity. We show that the results for Bernstein polynomials described above, extend to a wide class of linear approximators, which includes, but is much larger than, the classes of summability formulas discussed in [*3*] and [*4*].

We obtain all of the aforementioned results for functions which are convex with respect to an Extended Complete Tchebycheff system. This convexity will be defined at the end of the Introduction. We shall not give proofs of properties of such functions which will be used in the sequel. The reader is referred to [*6*] and [*12*] for a thorough discussion of properties of Extended Complete Tchebycheff systems and of convexity with respect to such systems.

In the last section we give several applications. These include the aforementioned results for Bernstein polynomials, some new theorems on approximation formulas involving convolutions of a distribution function, and an example involving the Weierstrass kernel on $(-\infty, \infty)$.

We proceed now with definitions of the basic concepts. Let (u_0, u_1) denote throughout the paper an Extended Complete Tchebycheff system (ECT-system) on $[a,b]$. With no loss of generality (see [*6*]) we may assume that $u_0(t) > 0$ on $[a,b]$ and that $u_1(t)$ can be represented in the form

$$u_1(t) = u_0(t) \int_a^t w_1(t)\, dt$$

where $w_1(t) > 0$ on $[a,b]$.

Definition 1. A linear combination of the form $c_0 u_0(t) + c_1 u_1(t)$ will be called a *first degree u-polynomial.*

Definition 2. A function $f(t)$ defined on (a,b) is said to be convex with respect to (u_0, u_1), provided

$$\begin{vmatrix} u_0(x_1) & u_0(x_2) & u_0(x_3) \\ u_1(x_1) & u_1(x_2) & u_1(x_3) \\ f(x_1) & f(x_2) & f(x_3) \end{vmatrix} \geqslant 0, \qquad a < x_1 < x_2 < x_3 < b. \tag{1}$$

Note that if such an $f(t)$ belongs to $C[a,b]$, then inequality (1) will hold, by continuity, for all $a \leqslant x_1 < x_2 < x_3 \leqslant b$.

The set of functions $f(t)$ satisfying (1) is denoted by $\mathscr{C}(u_0, u_1)$.

The case of ordinary convexity is obtained when we choose $u_i(t) \equiv t^i$, $i = 0, 1, \ldots$. A kth degree u-polynomial will then be an ordinary polynomial of degree k.

II. Direct Theorems

Let $[a,b]$ be a finite interval. We consider in this and the next section, real functionals and operators defined on $C[a,b]$—the space of functions continuous on $[a,b]$. The results can be readily extended, with almost identical proofs, to functionals and operators defined on suitable subsets of $C(a,b)$, $C[0,\infty)$ or $C(-\infty,\infty)$. These subsets are determined by imposing suitable order of growth conditions on the functions. Examples of such extensions, for the case of ordinary convexity, will be given in Section IV.

We recall the Riesz representation theorem (see e.g. [*10*]) which states that every real linear functional $\phi(f)$, defined on $C[a,b]$, can be represented by a signed measure of bounded variation $d\mu_\phi$, so that

$$\phi(f) = \int_a^b f(t)\, d\mu_\phi(t), \qquad \text{for all } f \in C[a,b] \tag{2}$$

and $\mu_\phi(a) = 0$. We note that $\phi(f)$ is a positive linear functional (p.l.f.) if and only if the associated measure $d\mu_\phi$ is a nonnegative measure.

LEMMA 1. *Let α be a point in $[a,b]$ and let $\phi(f)$ be a positive linear functional defined on $C[a,b]$ which satisfies the conditions*

$$\phi(u_0) = u_0(\alpha), \tag{3}$$

$$\phi(u_1) = u_1(\alpha). \tag{4}$$

If the support of the associated measure $d\mu_\phi$ is contained in $[\alpha,b]$ or in $[a,\alpha]$, then $d\mu_\phi$ must reduce to the Dirac measure with support at $t=\alpha$.

Proof. (I). Suppose that the support of $d\mu_\phi$ is contained in $[\alpha,b]$.

If $\alpha = b$, then in view of relation (3), the lemma is obviously true. Assume now that $\alpha \neq b$ and consider the function $g(t)$ defined by

$$g(t) = \begin{cases} 0, & a \leqslant t \leqslant \alpha, \\ u_1(t) - \dfrac{u_1(\alpha)}{u_0(\alpha)} u_0(t), & \alpha \leqslant t \leqslant b. \end{cases}$$

Since (u_0, u_1) is an ECT-system, it follows that $g(t)$ is a nonnegative function on $[a,b]$ which is strictly positive for $t > \alpha$. Thus, we have

$$\phi(g) \geqslant 0, \tag{5}$$

with strict inequality holding, unless the support of $d\mu_\phi$ consists of the point α alone.

On the other hand, the assumption on the support of $d\mu_\phi$, together with relations (3) and (4), yield

$$\begin{aligned} \phi(g) &= \phi\left[u_1(t) - \frac{u_1(\alpha)}{u_0(\alpha)} u_0(t)\right] = \phi(u_1) - \frac{u_1(\alpha)}{u_0(\alpha)} \phi(u_0) \\ &= u_1(\alpha) - \frac{u_1(\alpha)}{u_0(\alpha)} u_0(\alpha) = 0. \end{aligned}$$

Hence, the support of $d\mu_\phi$ must consist of the point α alone and in view of relation (3), the lemma is proved for this case.

(II). Suppose now that the support of $d\mu_\phi$ is contained in $[a,\alpha]$. If $\alpha = a$, the lemma is clearly implied by (3).

If $\alpha \neq a$, consider the function $h(t)$ defined by

$$h(t) = \begin{cases} \dfrac{u_1(\alpha)}{u_0(\alpha)} u_0(t) - u_1(t), & a \leqslant t \leqslant \alpha, \\ 0, & \alpha \leqslant t \leqslant b. \end{cases}$$

The rest of the proof proceeds with arguments identical with those used in the proof of case (I).

Lemma 1 yields immediately the following

COROLLARY 1. *Let $\phi(f)$ be a p.l.f. defined on $C[a,b]$, which satisfies*

$$\begin{cases}\phi(u_0) = u_0(a),\\ \phi(u_1) = u_1(a),\end{cases}$$

then $\phi(f) \equiv f(a)$, for all $f \in C[a,b]$.

We shall now prove the first "direct" theorem.

THEOREM 1. *Let α be a point in (a,b) and let $\phi(f)$ be a positive linear functional defined on $C[a,b]$, which satisfies (3) and (4). Then for any function $f(t)$ of $C[a,b] \cap \mathscr{C}(u_0,u_1)$ which does not coincide with a first-degree u-polynomial on any open interval containing α, we have*

$$\phi(f) \geqslant f(\alpha), \tag{6}$$

where equality holds if and only if

$$\phi(f) = f(\alpha), \qquad \textit{for every } f \in C[a,b]. \tag{7}$$

Proof. Let $f(t)$ be an arbitrary function of $C[a,b] \cap \mathscr{C}(u_0,u_1)$. We recall that a function which is convex with respect to an ECT-system (u_0,u_1), possesses a right derivative $f_R'(t)$ and a left derivative $f_L'(t)$ at all points of (a,b), and that these one-sided derivatives are equal almost everywhere. Furthermore, the functions

$$\frac{1}{w_1(t)}\left[\frac{f(t)}{u_0(t)}\right]_R', \qquad \frac{1}{w_1(t)}\left[\frac{f(t)}{u_0(t)}\right]_L',$$

where the subscripts R and L denote right and left derivatives, respectively, are monotone nondecreasing on (a,b) (see [6]).

We can, thus, choose a number c, such that

$$\begin{cases} c = \dfrac{1}{w_1(\alpha)}\left[\dfrac{f(\alpha)}{u_0(\alpha)}\right]', & \text{if } f'(\alpha) \text{ exists},\\[2ex] \dfrac{1}{w_1(\alpha)}\left[\dfrac{f(\alpha)}{u_0(\alpha)}\right]_L' < c < \dfrac{1}{w_1(\alpha)}\left[\dfrac{f(\alpha)}{u_0(\alpha)}\right]_R', & \text{otherwise}.\end{cases}$$

Then we have,

$$c \leqslant \frac{1}{w_1(t)}\left[\frac{f(t)}{u_0(t)}\right]_R', \qquad \text{for all } t > \alpha, \tag{8}$$

and

$$c \geqslant \frac{1}{w_1(t)}\left[\frac{f(t)}{u_0(t)}\right]_L', \qquad \text{for all } t < \alpha. \tag{9}$$

The assumption that $f(t)$ is not identical with a first-degree u-polynomial on any open interval containing α, implies that neither $\{[1/w_1(t)][f(t)/u_0(t)]_R'\}$ nor

$\{[1/w_1(t)][f(t)/u_0(t)]_L'\}$ can be constant on an open interval containing α. Hence, at least one of the inequalities (8)–(9) is a strict inequality.

Consider now the function

$$l(t) = c\left[u_1(t) - \frac{u_1(\alpha)}{u_0(\alpha)}u_0(t)\right] + \frac{f(\alpha)}{u_0(\alpha)}u_0(t). \tag{10}$$

Relations (3) and (4), on conjunction with the linearity of $\phi(f)$, yield

$$\begin{aligned}\phi(l) &= c\left[\phi(u_1) - \frac{u_1(\alpha)}{u_0(\alpha)}\phi(u_0)\right] + \frac{f(\alpha)}{u_0(\alpha)}\phi(u_0)\\ &= c\left[u_1(\alpha) - \frac{u_1(\alpha)}{u_0(\alpha)}u_0(\alpha)\right] + \frac{f(\alpha)}{u_0(\alpha)}u_0(\alpha)\\ &= f(\alpha). \end{aligned}\tag{11}$$

Observe, next, that relation (8) and the definition of $w_1(t)$, yield, for $t > \alpha$, the following result:

$$\begin{aligned} f(t) - \frac{f(\alpha)}{u_0(\alpha)}u_0(t) &= u_0(t)\left[\frac{f(t)}{u_0(t)} - \frac{f(\alpha)}{u_0(\alpha)}\right]\\ &= u_0(t)\int_\alpha^t \left[\frac{f(t)}{u_0(t)}\right]_R' dt = u_0(t)\int_\alpha^t \frac{1}{w_1(t)}\left[\frac{f(t)}{u_0(t)}\right]_R' w_1(t)\,dt\\ &\geqslant u_0(t)\int_\alpha^t c w_1(t)\,dt\\ &= cu_0(t)\left[\int_a^t w_1(t)\,dt - \int_a^\alpha w_1(t)\,dt\right]\\ &= c\left[u_0(t)\int_a^t w_1(t)\,dt - \frac{u_0(t)}{u_0(\alpha)}u_0(\alpha)\int_a^\alpha w_1(t)\,dt\right]\\ &= c\left[u_1(t) - \frac{u_1(\alpha)}{u_0(\alpha)}u_0(t)\right]. \end{aligned}$$

Performing a similar computation with the aid of relation (9) for $t < \alpha$, we find

$$\frac{f(\alpha)}{u_0(\alpha)}u_0(t) - f(t) \leqslant c\left[\frac{u_1(\alpha)}{u_0(\alpha)}u_0(t) - u_1(t)\right], \qquad \text{for all } t < \alpha.$$

Combining both inequalities, we obtain

$$f(t) - \frac{f(\alpha)}{u_0(\alpha)}u_0(t) \geqslant c\left[u_1(t) - \frac{u_1(\alpha)}{u_0(\alpha)}u_0(t)\right], \qquad \text{for all } t \neq \alpha,\ t \in [a,b].$$

Since both sides are zero for $t = \alpha$, we find, using the definition (10), that

$$f(t) \geqslant l(t), \qquad \text{for all } t \in [a,b]. \tag{12}$$

Relations (12) and (11) imply that

$$\phi(f) \geqslant \phi(l) = f(\alpha). \tag{13}$$

Thus, the first statement of the theorem has been proved, and to complete the proof, we need only examine the circumstances under which equality holds.

Tracing the inequalities which led to (12), we see that the remarks following relation (9) imply that at least one of the following strict inequalities holds:

$$f(t) > l(t), \qquad \text{for all } t > \alpha, \tag{14}$$

or,

$$f(t) > l(t), \qquad \text{for all } t < \alpha. \tag{15}$$

We assume now that equality holds in (13), and distinguish between the following two cases:

(I). Suppose that (14) holds. Consider the function

$$g(t) = \begin{cases} 0, & a \leqslant t \leqslant \alpha, \\ f(t) - l(t), & \alpha \leqslant t \leqslant b. \end{cases}$$

In view of (12) and the assumption of equality in (13), we have

$$0 = \phi(f - l) \geqslant \phi(g). \tag{16}$$

Since $g(t)$ is a nonnegative function which is strictly positive for $t > \alpha$, and since ϕ is a p.l.f., we deduce that equality holds in (16) and that the associated measure $d\mu_\phi$ can have no mass on $(\alpha, b]$. This is equivalent to the statement that the support of $d\mu_\phi$ is contained in $[a, \alpha]$. Noting that (3) and (4) are satisfied, and appealing to Lemma 1, we deduce that $d\mu_\phi$ reduces to the Dirac measure with support at $t = \alpha$. This is equivalent to (7).

(II). Suppose that (15) holds. A similar discussion involving the function

$$h(t) = \begin{cases} f(t) - l(t), & a \leqslant t \leqslant \alpha, \\ 0, & \alpha \leqslant t \leqslant b, \end{cases}$$

and invoking Lemma 1, proves that equality in (13) implies (7), in this case as well. This concludes the proof of Theorem 1.

Remark. Inequality (6) may be proved also by using the concept of dual convexity cones. Specifically, one can use Theorem B of [*13*] (which is essentially contained, albeit in a weaker form which is insufficient for our purpose, in [*5*]) to obtain the inequality. However, it is much harder to analyze the circumstances under which equality holds. We preferred the proof presented here, since the analysis of the circumstances of equality plays an essential role in the converse theorems.

Consider now a positive linear operator (p.l.o.) transforming a function $f \in C[a,b]$ into a function defined on $[c,d]$, $a \leqslant c < d \leqslant b$.

Noting that the restriction of such an operator, obtained by fixing a point $x = \alpha$, is a positive linear functional, the following theorem can be easily derived from Theorem 1 and Corollary 1.

THEOREM 2. *Let $L(f;x)$ be a positive linear operator defined on $C[a,b]$, which satisfies the conditions*

$$L(u_0;x) \equiv u_0(x), \qquad x \in [c,d], \tag{17}$$

$$L(u_1;x) \equiv u_1(x), \qquad x \in [c,d]. \tag{18}$$

Then, for every function $f \in C[a,b] \cap \mathscr{C}(u_0,u_1)$, we have

$$L(f;x) \geqslant f(x), \qquad \text{for } c \leqslant x \leqslant d, \tag{19}$$

and

$$L(f;a) = f(a), \qquad L(f;b) = f(b), \tag{20}$$

whenever $a = c$ and $b = d$.

Equality sign in (19) can be achieved at a point $x = \alpha$, $c < \alpha < d$, only if either $f(t)$ coincides with a first degree u-polynomial on some open interval containing α, or $L(f;\alpha) \equiv f(\alpha)$, for all $f \in C[a,b]$.

We shall now show that conditions (17) and (18) are indispensable; explicitly, we shall prove

THEOREM 3. *Let $L(f;x)$ be a positive linear operator such that for every function $f(t) \in C[a,b] \cap \mathscr{C}(u_0,u_1)$, inequality* (19) *holds. Then* (17) *and* (18) *have to be satisfied.*

Proof. Consider the functions $u_0(t)$ and $-u_0(t)$. Both belong to

$$C[a,b] \cap \mathscr{C}(u_0,u_1),$$

so that (19) implies

$$L(u_0;x) \geqslant u_0(x),$$

$$-L(u_0;x) = L(-u_0;x) \geqslant -u_0(x).$$

These inequalities imply (17). Similarly, the necessity of (18) is implied by the fact that both $u_1(t)$ and $-u_1(t)$ belong to $C[a,b] \cap \mathscr{C}(u_0,u_1)$.

Remark. Let $C[a,b]$ be replaced by $C(I)$, where I is any real interval (finite or infinite, closed or open). Consider a subclass $\mathscr{F}$ of $C(I)$. Suppose (u_0,u_1) is an ECT-system on I such that $u_i \in \mathscr{F}$, $i = 1, 2$, and define convexity with respect to (u_0,u_1) as in Definition 2.

ASSERTION. In such a set-up, Lemma 1 and Theorems 1–3 are satisfied, when the functionals and the operators are defined on $\mathscr{F}$.

This assertion follows by a verbatim retracing of the proofs of Lemma 1 and Theorems 1–3. We shall show in Section IV, that the theorems for the approximation formulas discussed in [3] and [4], are covered by this assertion.

III. CONVERSE THEOREMS

We shall show in this section that convexity properties of a function can be deduced from the behaviour of its approximations by means of positive linear operators.

We start by introducing some definitions.

DEFINITION 3. A sequence of positive linear functionals $\{\phi_n(f)\}_{n=1}^{\infty}$, defined on $C[a,b]$, is said to be *strongly centered* at the point α, $a < \alpha < b$, if for every fixed pair of values $\eta > \delta > 0$ such that $a \leqslant \alpha \pm \eta \leqslant b$, we have

$$\lim_{n\to\infty} \frac{\mu_n[\alpha+\eta, b]}{\mu_n[\alpha+\delta, \alpha+\eta]} = 0, \qquad \lim_{n\to\infty} \frac{\mu_n[a, \alpha-\eta]}{\mu_n[\alpha-\eta, \alpha-\delta]} = 0, \tag{21}$$

where $d\mu_n(t)$ is the Riesz measure associated with $\phi_n(f)$.

It is implicitly assumed in the definition that the quotients are meaningful for large enough n, e.g., that $\mu_n[\alpha+\delta, \alpha+\eta] > 0$ for $n > N$, say.

DEFINITION 4. Let $\{L_n(f;x)\}_{n=1}^{\infty}$ be a sequence of positive linear operators transforming functions $f \in C[a,b]$ into functions defined on $[c,d]$, $a \leqslant c < d \leqslant b$. The sequence is said to be *strongly centered* on (c,d), if for each fixed z, $c < z < d$, the sequence of p.l.f's $\{L_n(f;z)\}_{n=1}^{\infty}$ is strongly centered at z.

The requirement that a sequence of p.l.f's be strongly centered at α is a strong requirement. For example, as we shall see later (cf. Lemma 3), it essentially ensures that $\phi_n(f) \to f(\alpha)$ for all $f \in C[a,b]$.

We come now to the main theorem of this section.

THEOREM 4. *Let $\{L_n(f;x)\}_{n=1}^{\infty}$ be a sequence of positive linear operators which is strongly centered on (c,d), satisfying*

$$L_n(u_i;x) \equiv u_i(x), \qquad c \leqslant x \leqslant d;\ i = 0, 1;\ n = 1, 2, \ldots. \tag{22}$$

If, for a function $f(t) \in C[a,b]$, the inequality

$$L_n(f;x) \geqslant f(x), \qquad \textit{for all } x \in (c,d) \tag{23}$$

is valid for all n, then $f(t)$ belongs to $\mathscr{C}(u_0, u_1)$ on $[c,d]$.

Proof. Suppose that conditions (22) are satisfied and that $f(x)$ does not belong to $\mathscr{C}(u_0,u_1)$ on $[c,d]$. We shall prove that these assumptions imply that (23) cannot be valid.

Since $f(x)$ does not belong to $\mathscr{C}(u_0,u_1)$, there exist three points

$$c \leqslant x_1 < x_2 < x_3 \leqslant d,$$

such that

$$\begin{vmatrix} u_0(x_1) & u_0(x_2) & u_0(x_3) \\ u_1(x_1) & u_1(x_2) & u_1(x_3) \\ f(x_1) & f(x_2) & f(x_3) \end{vmatrix} < 0. \tag{24}$$

Define the function $g(x)$ on $[a,b]$ by

$$g(x) = \frac{1}{\begin{vmatrix} u_0(x_1) & u_0(x_3) \\ u_1(x_1) & u_1(x_3) \end{vmatrix}} \begin{vmatrix} u_0(x_1) & u_0(x_3) & u_0(x) \\ u_1(x_1) & u_1(x_3) & u_1(x) \\ f(x_1) & f(x_3) & f(x) \end{vmatrix}. \tag{25}$$

Note that the denominator is strictly positive, since (u_0,u_1) is an ECT-system. By expanding the determinant by its last column, we see that $g(x)$ can be expressed as

$$g(x) = f(x) - \alpha u_1(x) - \beta u_0(x), \tag{26}$$

where α and β are constants (depending on x_1, x_3, u_0, u_1 and f). In terms of $g(x)$, relation (24) is equivalent to

$$g(x_2) > 0, \tag{27}$$

and relation (25) implies that

$$g(x_1) = g(x_3) = 0. \tag{28}$$

Set $M = \max_{x \in [x_1, x_3]} g(x)/u_0(x)$. Relations (27), (28) imply that $M > 0$ and that it is attained only at interior points of $[x_1,x_3]$. Set

$$y = \max\{x;\, x \in [x_1,x_3],\, g(x) = M\}, \qquad z = \min\{x;\, x \in [x_1,x_3],\, g(x) = M\}.$$

Relation (28) implies that $x_1 < z \leqslant y < x_3$.

Define now the first-degree u-polynomial $l(x)$ by

$$l(x) = \alpha u_1(x) + \beta u_0(x) + M u_0(x). \tag{29}$$

In view of the definitions of $g(x)$, M, y and z, the following relations hold:

$$f(y) = l(y), \tag{30}$$

$$f(x) < l(x), \qquad \text{for } x \in [x_1, z) \cup (y, x_3], \tag{31}$$

$$f(x) \leqslant l(x), \qquad \text{for } x \in [x_1, x_3]. \tag{32}$$

Set

$$\left.\begin{aligned}
&M_0 = \max_{t \in [a,b]} |l(t) - f(t)| \\
&\eta_1 = y - x_1, \qquad \eta_2 = x_3 - y, \qquad \delta_1 = y - \frac{x_1 + z}{2}, \qquad \delta_2 = \frac{x_3 + y}{2} - y, \\
&m_0 = \min(l(t) - f(t)), \qquad \text{for } t \in \left[x_1, \frac{x_1 + z}{2}\right] \cup \left[\frac{x_3 + y}{2}, x_3\right].
\end{aligned}\right\} \tag{33}$$

Observe that $\eta_1 > \delta_1 > 0$, $\eta_2 > \delta_2 > 0$, and that the sequence $\{L_n(f; y)\}_{n=1}^{\infty}$ is strongly centered at y. Thus, (21) holds for both pairs of (δ, η). Choose $\epsilon = (m_0/2M_0)$, and note that $\epsilon > 0$ by virtue of the continuity of $l(t) - f(t)$ and relation (31). By appealing to (21), and setting $d\bar{\mu}_n(t) = d\mu_n(y; t)$, we deduce that there exists an N such that

$$\left.\begin{aligned}
\bar{\mu}_n[y + \eta_2, b] &< (m_0/2M_0)\, \bar{\mu}_n[y + \delta_2, y + \eta_2], \qquad \text{for } n > N, \\
\bar{\mu}_n[a, y - \eta_1] &< (m_0/2M_0)\, \bar{\mu}_n[y - \eta_1, y - \delta_1], \qquad \text{for } n > N.
\end{aligned}\right\} \tag{34}$$

Using now relations (32)–(34), we have the following chain of inequalities for $n > N$,

$$\begin{aligned}
L_n(l - f; y) &= \int_a^b (l - f)\, d\bar{\mu}_n(t) \\
&\geqslant \int_{x_1}^{x_3} (l - f)\, d\bar{\mu}_n(t) - M_0 \int_a^{x_1} d\bar{\mu}_n(t) - M_0 \int_{x_3}^b d\bar{\mu}_n(t) \\
&\geqslant \int_{x_1}^{(x_1+z)/2} (l - f)\, d\bar{\mu}_n(t) + \int_{(x_3+y)/2}^{x_3} (l - f)\, d\bar{\mu}_n(t) \\
&\qquad - M_0\{\bar{\mu}_n[a, y - \eta_1] + \bar{\mu}_n[y + \eta_2, b]\} \\
&\geqslant m_0\{\bar{\mu}_n[y - \eta_1, y - \delta_1] + \bar{\mu}_n[y + \delta_2, y + \eta_2] \\
&\qquad - M_0\{\bar{\mu}_n[a, y - \eta_1] + \bar{\mu}_n[y + \eta_2, b]\} \\
&> M_0\{\bar{\mu}_n[a, y - \eta_1] + \bar{\mu}_n[y + \eta_2, b]\} \geqslant 0.
\end{aligned}$$

Thus, we have

$$L_n(f; y) < L_n(l; y), \qquad \text{for } n > N.$$

Since $l(x)$ is a first-degree u-polynomial, (22) implies that $L_n(l; y) = l(y)$, $n = 1, 2, \ldots$. Using (30), we obtain

$$L_n(f; y) < f(y), \qquad \text{for } n > N,$$

which clearly demonstrates that (23) does not hold. This completes the proof of the theorem.

We prove, next, the following lemma, showing an implication of the assumption of strong centeredness.

LEMMA 3. *Let a sequence of positive linear functionals $\{\phi_n(f)\}_{n=1}^{\infty}$ be strongly centered at α, and let it satisfy*

$$\lim_{n\to\infty} \phi_n(u_0) = u_0(\alpha). \tag{35}$$

Then

$$\lim_{n\to\infty} \phi_n(f) = f(\alpha), \qquad \textit{for all } f \in C[a,b]. \tag{36}$$

Proof. Note, first, that (35) implies that there exists a K such that $|\phi_n(u_0)| < K$, $n = 1, 2, \ldots$. Set now $k = \min_{t\in[a,b]} u_0(t)$. Then $k > 0$, and we have

$$\mu_n[a,b] = \int_a^b d\mu_n(t) \leqslant (1/k) \int_a^b u_0(t)\, d\mu_n(t) \leqslant (K/k), \qquad n = 1, 2, \ldots, \tag{37}$$

where $d\mu_n(t)$ is the Riesz measure associated with $\phi_n(f)$.

Let now $\eta > 0$, such that $a \leqslant \alpha \pm \eta \leqslant b$, be given. Then

$$\lim_{n\to\infty} \{\mu_n[\alpha+\eta, b] + \mu_n[a, \alpha-\eta]\} = 0. \tag{38}$$

Indeed, (37) implies that, for all n,

$$\begin{aligned}\{\mu_n[\alpha+\eta, b] + \mu_n[a, \alpha-\eta]\} &\leqslant (K/k)\left\{\frac{\mu_n[\alpha+\eta, b] + \mu_n[a, \alpha-\eta]}{\mu_n[a,b]}\right\} \\ &\leqslant (K/k)\left\{\frac{\mu_n[\alpha+\eta, b]}{\mu_n[\alpha+\delta, \alpha+\eta]} + \frac{\mu_n[a, \alpha-\eta]}{\mu_n[\alpha-\eta, \alpha-\delta]}\right\},\end{aligned}$$

where $\eta > \delta > 0$. Since the last expression tends to zero by virtue of (21), relation (38) is established.

Let now an $f \in C[a,b]$ and an $\epsilon > 0$ be given. By uniform continuity, we may choose an $\eta > 0$ such that

$$\left| f(t) - \frac{f(\alpha)}{u_0(\alpha)} u_0(t) \right| < \frac{\epsilon k}{3K}, \qquad \text{if } |t-\alpha| < \eta. \tag{39}$$

Using, next, relations (35) and (38), we choose an N_0 such that, for $n > N_0$ we have,

$$\left| \frac{f(\alpha)\,[\phi_n(u_0) - u_0(\alpha)]}{u_0(\alpha)} \right| < \epsilon/3, \qquad \mu_n[\alpha+\eta, b] + \mu_n[a, \alpha-\eta] < (\epsilon/3)(1/M_0), \tag{40}$$

where
$$M_0 = \max_{t\in[a,b]} |f(t) - f(\alpha)\, u_0(t)/u_0(\alpha)|.$$

Computing now, for $n > N_0$, the difference $|\phi_n(f) - f(\alpha)|$, we find, by using (37), (39) and (40), that

$$
\begin{aligned}
|\phi_n(f)-f(\alpha)| &= \left|\phi_n(f)-\frac{f(\alpha)}{u_0(\alpha)}\phi_n(u_0)+\frac{f(\alpha)}{u_0(\alpha)}\phi_n(u_0)-f(\alpha)\right| \\
&\leqslant \frac{\epsilon}{3}+\left|\phi_n\left[f(t)-\frac{f(\alpha)}{u_0(\alpha)}u_0(t)\right]\right| \\
&\leqslant \frac{\epsilon}{3}+\frac{\epsilon k}{3K}\int_{|t-\alpha|<\eta} d\mu_n(t)+M_0\left[\int_{\alpha+\eta}^{b} d\mu_n(t)+\int_a^{\alpha-\eta} d\mu_n(t)\right] \\
&\leqslant \frac{\epsilon}{3}+\frac{\epsilon k}{3K}\int_a^b d\mu_n(t)+\frac{\epsilon}{3} \\
&\leqslant \epsilon.
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, this proves the lemma.

THEOREM 5. *Let $\{L_n(f;x)\}_{n=1}^{\infty}$ be a sequence of positive linear operators which is strongly centered on (c,d) and satisfying (22). If, for a function $f(t) \in C[a,b]$, the inequality*

$$L_n(f;x) \geqslant L_{n+1}(f;x), \qquad n=1,2,\ldots, \tag{41}$$

is satisfied for each $x \in (c,d)$, then $f(t)$ belongs to $\mathscr{C}(u_0,u_1)$ on $[c,d]$.

Proof. Since $\{L_n(f;x)\}_{n=1}^{\infty}$ is strongly centered on (c,d), Lemma 3 implies that

$$\lim_{n\to\infty} L_n(f;x) = f(x), \qquad \text{for all } x \in (c,d). \tag{42}$$

We shall show that from (42) and (41) it follows that

$$L_n(f;x) \geqslant f(x), \qquad \text{for all } x \in (c,d),\ n=1,2,\ldots. \tag{43}$$

Indeed, if for some point $x_0 \in (c,d)$ and some natural number N, the reverse inequality holds, i.e.,

$$L_N(f;x_0) < f(x_0),$$

then, by (41), for all $n \geqslant N$ we have

$$L_n(f;x_0) \leqslant L_N(f;x_0) < f(x_0),$$

which is impossible, by virtue of (42).

Hence, relation (43) holds, and the theorem follows by appealing to Theorem 4.

Remark 1. There exist examples of sequences of positive linear operators $\{L_n(f;x)\}_{n=1}^{\infty}$ converging uniformly to $f(x)$ on $[a,b]$ for which the "converse" theorem does not hold.

Remark 2. Note that all the results of this section remain valid for operators and functionals defined on the space of functions which are continuous and bounded on the open interval (a,b).

IV. Applications

In this section we shall present several applications of the foregoing analysis. These include the results on Bernstein polynomials mentioned in Section I and some general, new theorems involving operators which arise out of Probability Theory considerations. An example involving the Weierstrass kernel on $(-\infty,\infty)$ is also discussed.

1. Bernstein Polynomials.

The nth order Bernstein polynomial $B_n(f;x)$ is defined, for $f \in C[0,1]$ by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}, \qquad 0 \leqslant x \leqslant 1;\, n = 1, 2, \ldots. \tag{44}$$

The sequence $\{B_n(f;x)\}_{n=1}^{\infty}$ is clearly a sequence of p.l.o's. Consider the ECT-system $(1,t)$ and note that a simple computation yields

$$\begin{aligned} B_n(1,x) &\equiv 1, \qquad n = 1, 2, \ldots, \\ B_n(t;x) &\equiv x, \qquad n = 1, 2, \ldots. \end{aligned} \tag{45}$$

We shall now prove that the sequence $\{B_n(f;x)\}_{n=1}^{\infty}$ is strongly centered on $(0,1)$. Indeed, let z be a point in $(0,1)$. The Riesz measure $d\mu_n(t)$ corresponding to $B_n(f;z)$ is a discrete measure with $n+1$ atoms of mass at the points i/n, $i = 0, 1, \ldots, n$. The mass at i/n is given by

$$\binom{n}{i} z^i(1-z)^{n-i} = b(i;n,z).$$

Hence, strong centeredness of $\{B_n(f;z)\}_{n=1}^{\infty}$ at z is equivalent to the following statement:

For all $\eta > \delta > 0$ such that $0 < z \pm \eta < 1$, we have

$$\lim_{n\to\infty} \frac{\sum_{i/n \geqslant z+\eta} b(i;n,z)}{\sum_{z+\eta \geqslant i/n \geqslant z+\delta} b(i;n,z)} = 0, \qquad \lim_{n\to\infty} \frac{\sum_{i/n \leqslant z-\eta} b(i;n,z)}{\sum_{z-\eta \leqslant i/n \leqslant z-\delta} b(i;n,z)} = 0 \tag{46}$$

We shall prove the relation on the right. The other relation follows in precisely the same way. Denote the quotient on the right by $\beta(n;\eta,\delta)$. The following inequality holds for the binomial coefficients (see [2], Vol. 1, p. 141, Theorem 2):

$$\sum_{k=0}^{s} b(k;n,z) \leqslant b(s;n,z)\cdot\frac{(n-s+1)}{(n+1)z-s}, \qquad \text{for } s < nz.$$

Taking $s = [nz - \eta n]$, (where $[x]$ denotes, as usual, the integral part of x), we thus have for the numerator of $\beta(n;\eta,\delta)$, the estimate

$$\begin{aligned}\sum_{i/n \leqslant z-\eta} b(i;n,z) &\leqslant b([nz-\eta n];n,z)\cdot\frac{(n-[nz-\eta n]+1)\,z}{(n+1)\,z-[nz-\eta n]}\\ &\leqslant b([nz-\eta n];n,z)\cdot\frac{[n+2-nz+\eta n]\,z}{\eta n+z}\\ &\leqslant Cb([nz-\eta n];n,z),\end{aligned}$$

where C is a positive constant, not depending on n.

On the other hand, using the fact that the binomial coefficients $b(k;n,z)$ are nondecreasing in k for $0 < k < nz$, we find

$$\begin{aligned}\sum_{z-\eta \leqslant i/n \geqslant z-\delta} b(i;n,z) &\geqslant ([zn-\delta n]-[zn-\eta n])\, b([nz-\eta n];n,z)\\ &\geqslant [n(\eta-\delta)-1]\, b([nz-\eta n];n,z).\end{aligned}$$

Combining the two estimates, we obtain

$$\beta(n;\eta,\delta) \leqslant \frac{C}{n(\eta-\delta)-1}, \qquad n = 1, 2, \ldots.$$

This clearly implies that

$$\lim_{n\to\infty} \beta(n;\eta,\delta) = 0,$$

proving the right-hand side of (46).

This establishes the strong centeredness of the sequence $\{B_n(f;x)\}_{n=1}^{\infty}$ on $(0,1)$. Thus, we may apply Theorems 2, 4 and 5 to deduce:

(a) A necessary and sufficient condition for a function $f(x) \in C[0,1]$ to be convex on $[0,1]$, is that $B_n(f;x) \geqslant f(x)$ for all n and all $x \in (0,1)$.

(b) If $B_n(f;x) \geqslant B_{n+1}(f;x)$ for all n and all $x \in (0,1)$, and $f(x) \in C[0,1]$, then $f(x)$ is convex on $[0,1]$.

We recall here, for the sake of completeness, that the condition in (b) is also necessary (see Section I).

2. Approximations on a Finite Interval Involving Convolutions.

Let $G(t)$ be a distribution function of a positive random variable X which takes values only on $[0,b]$, $b > 1$. Let it be so normalized, that

$$E[X] \equiv \int_0^b t\,dG(t) = 1, \tag{47}$$

i.e., the expectation of the random variable is 1.

Define a squence of positive linear operators $\{U_n(f;x)\}_{n=1}^{\infty}$, by

$$U_n(f;x) = \int_0^{nb} f\left(\frac{tx}{n}\right) dG^{(n)}(t), \qquad n = 1, 2, \ldots, \tag{48}$$

where $G^{(n)}(t)$ is the n-fold convolution of $G(t)$. These operators are defined on $C[0,b]$, transforming a function of $C[0,b]$ into a function defined on $[0,1]$.

Consider the ECT-system $(1,t)$. By virtue of $G^{(n)}(t)$ being a distribution function, it follows that

$$U_n(1;x) \equiv \int_0^{nb} dG^{(n)}(t) \equiv 1, \qquad n = 1, 2, \ldots. \tag{49}$$

We now make use of the following standard result from Probability Theory (see e.g. [2], Vol. 2): Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed, random variables with a common distribution function $G(t)$, and let $E[X_i]$, $i = 1, 2, \ldots, n$, denote the expectation of the ith random variable. Then

$$nE[X] = \sum_{i=1}^{n} E[X_i] = E\left[\sum_{i=1}^{n} X_i\right] = \int_0^{nb} t\, dG^{(n)}(t).$$

Since in the case under consideration, $E[X] = 1$, we have

$$U_n(t;x) = \int_0^{nb} (xt/n)\, dG^{(n)}(t) \equiv x, \qquad n = 1, 2, \ldots. \tag{50}$$

LEMMA 4. *With the above definitions, if X is nondegenerate, then $\{U_n(f;x)\}_{n=1}^{\infty}$ is strongly centered on* $(0,1)$.

Proof. Let z be a fixed point, $0 < z < 1$. Then

$$U_n(f;z) = \int_0^{nb} f(tz/n)\, dG^{(n)}(t) = \int_0^{bz} f(t)\, dG^{(n)}(tn/z).$$

Thus, $d\bar{\mu}_n(t) = d\mu_n(z;t) = dG^{(n)}(tn/z)$, and it is defined on $[0,bz]$.

Strong centeredness of the sequence $\{U_n(f;z)\}_{n=1}^{\infty}$ at z will thus be proved, if we show that for all $\eta > \delta > 0$ such that $0 \leqslant 1 \pm \eta \leqslant b$, the limit relations

$$\lim_{n\to\infty} \frac{\bar{\mu}_n[z+\eta z, bz]}{\bar{\mu}_n[z+\delta z,\, z+\eta z]} = 0, \qquad \lim_{n\to\infty} \frac{\bar{\mu}_n[0, z-\eta z]}{\bar{\mu}_n[z-\eta z,\, z-\delta z]} = 0, \tag{51}$$

hold.

We shall prove the relation on the left. The other one follows in precisely the same way.

Denoting the quotient on the left by $\beta(n;\delta,\eta)$, and making obvious changes of variable, we find

$$\beta[n;\delta,\eta] = \frac{\bar{\mu}_n[z+\eta z, bz]}{\bar{\mu}_n[z+\delta z,\, z+\eta z]} = \frac{\int_{(1+\eta)z}^{bz} dG^{(n)}(tn/z)}{\int_{(1+\delta)z}^{(1+\eta)z} dG^{(n)}(tn/z)} = \frac{\int_{(1+\eta)n}^{bn} dG^{(n)}(t)}{\int_{(1+\delta)n}^{(1+\eta)n} dG^{(n)}(t)}.$$

Employing probabilistic interpretations, and denoting $\sum_{i=1}^{n} X_i$ by S_n, we have

$$\beta[n;\delta,\eta] = \frac{\Pr\{n(1+\eta) \leqslant S_n\}}{\Pr\{n(1+\delta) \leqslant S_n \leqslant n(1+\eta)\}}. \tag{52}$$

In order to prove that $\beta(n;\delta,n) \to 0$, we make use of the following recent results of Petrov [9] (adapted for the case of a bounded random variable):

Set, for $h > 0$,

$$\left.\begin{aligned} R(h) &= \int_a^b e^{ht}\, dG(t), \\ m(h) &= \frac{1}{R(h)} \int_a^b t\, e^{ht}\, dG(t), \\ \sigma^2(h) &= \frac{dm(h)}{dh}. \end{aligned}\right\} \tag{53}$$

We remark that $\lim_{h\to 0} m(h) = E(X)$, $\lim_{h\to\infty} m(h) = b$, and $m(h)$ is a strictly increasing function.

THEOREM A [9]. *Let $X_1, X_2, \ldots,$ be a sequence of independent random variables having the same nonlattice distribution $G(t)$. Let c be any constant satisfying $E[X] < c < b$. Then*:

$$\Pr\{S_n \geqslant nc\} = \frac{\exp\{n[\log R(h) - hc]\}}{h\sigma(h)\,(2\pi n)^{1/2}}\,(1 + o(1))$$

as $n \to \infty$. Here h is the unique real root of the equation $m(h) = c$.

THEOREM B [9]. *Let $X_1, X_2, \ldots,$ be a sequence of independent random variables having the same lattice distribution $G(t)$ (such that only values of the form $a + kH$, $k = 0, \pm 1, \pm 2, \ldots$ are taken with positive probability). Let c be any constant satisfying $E[X] < c < b$. Then*

$$\Pr\{S_n \geqslant nc\} = \frac{H \exp\{n[\log R(h) - hc]\}}{\sigma(h)\,(1 - e^{hH})\,(2\pi n)^{1/2}} \left(1 + O\left(\frac{1}{n}\right)\right)$$

as $n \to \infty$. Here h is the unique real root of the equation $m(h) = c$.

We remark that results of a similar nature have been obtained in [*1*].

Returning to the case under consideration, we observe that $E[X] = 1$, and that $\beta[n;\delta,\eta] \to 0$ as $n \to \infty$, iff

$$\alpha[n;\delta,\eta] = \frac{\Pr\{n(1+\eta) \leqslant S_n\}}{\Pr\{n(1+\delta) \leqslant S_n\}} \to 0 \qquad \text{as } n \to \infty. \tag{54}$$

Appealing to Theorems A and B, we deduce that there exists an N, such that

$$\alpha[n;\delta,\eta] \leqslant 4\frac{\exp\{n[\log R(h_2) - h_2(1+\eta)]\}}{\exp\{n[\log R(h_1) - h_1(1+\delta)]\}}\cdot\frac{u(h_1)}{u(h_2)}, \qquad \text{for } n > N, \tag{55}$$

where $u(h) = h\sigma(h)$ for the case of a nonlattice distribution, and

$$u(h) = \sigma(h)(1 - e^{hH})$$

for the case of a lattice distribution. Here h_2 and h_1 are defined by:

$$\left.\begin{aligned} m(h_1) &= 1+\delta, \\ m(h_2) &= 1+\eta. \end{aligned}\right\} \tag{56}$$

Set

$$Q(x) = \log R(h) - hx, \tag{57}$$

where h is defined by $m(h) = x$.

Then using (55), (56), and the fact that $u(h_1)$ and $u(h_2)$ are fixed, (54) will follow if we prove that $Q(x)$ is a strictly decreasing function for $x > 1$.

Computing the derivative of $Q(x)$, we obtain

$$\begin{aligned} Q'(x) &= \frac{1}{R(h)} R'(h)\cdot\frac{dh}{dx} - x\frac{dh}{dx} - h \\ &= [m(h) - x]\frac{dh}{dx} - h = -h < 0. \end{aligned}$$

This proves that the right-hand side of (55) tends to zero as n tends to infinity, and consequently so do $\alpha[n;\delta,\eta]$ and $\beta[n;\delta,\eta]$. This demonstrates that the sequence $\{U_n(f,z)\}_{n=1}^{\infty}$ is strongly centered at z. Since z was an arbitrary point of $(0,1)$, the proof of the lemma is complete.

Having Lemma 4 as well as relations (49), (50) at our disposal, we can apply theorems 2, 4 and 5 to deduce:

THEOREM 6. *Let $G(t)$ be a distribution function of a nondegenerate positive random variable which takes values on* $[0,b]$, *and let it be normalized by* (47). *Let the operators $U_n(f;x)$, $n = 1, 2, \ldots,$ be defined on $C[0,b]$ by* (48). *Then*

(a) *A necessary and sufficient condition for a function of $C[0,b]$ to be convex on* $[0,1]$, *is that $U_n(f;x) \geqslant f(x)$, for all n and all $x \in [0,1]$.*

(b) *If $U_n(f;x) \geqslant U_{n+1}(f;x)$ for all n and all $x \in [0,1]$, and if $f(x) \in C[0,b]$, then $f(x)$ is convex on* $[0,1]$.

Remark. The condition formulated in (b) is also a necessary condition for the convexity of $f(x)$. This follows from a result due to Marshall and Proschan (see [*4*], Theorem 3.8).

We may obtain similar results by using other approximation formulas, defined as follows:

Let $\{G_x(t), 0 < x < 1\}$ denote a family of distribution functions of positive random variables taking values on $[0, 1/x]$, such that

$$\int_0^{1/x} t\, dG_x(t) = 1. \tag{58}$$

Define a sequence of positive linear operators $\{V_n(f;x)\}_{n=1}^{\infty}$, by

$$V_n(f;x) = \int_0^{n/x} f\left(\frac{tx}{n}\right) dG_x^{(n)}(t), \qquad n = 1, 2, \ldots; 0 < x < 1, \tag{59}$$

and

$$V_n(f;0) = f(0), \qquad V_n(f;1) = f(1), \qquad n = 1, 2, \ldots. \tag{60}$$

These operators are defined on $C[0,1]$, transforming a function of $C[0,1]$ into a function defined on $[0,1]$.

The theory developed in this subsection for the operators $U_n(f;x)$, $n = 1$, $2, \ldots$, goes over with no changes, and we have (incorporating the result quoted in the remark following Theorem 6)

THEOREM 7. *Let $\{G_x(t), 0 < x < 1\}$ be a family of distribution functions of nondegenerate positive random variables which take values on $[0, 1/x]$ and are normalized by* (58). *Let the operators $V_n(f;x)$, $n = 1, 2, \ldots$, be defined on $C[0,1]$ by* (59)–(60). *Then*

(a) *A necessary and sufficient condition for a function $f \in C[0,1]$ to be convex on $[0,1]$, is that $V_n(f;x) \geqslant f(x)$, for all n and all $x \in [0,1]$.*

(b) *A necessary and sufficient condition for a function $f \in C[0,1]$ to be convex on $[0,1]$, is that $V_n(f;x) \geqslant V_{n+1}(f;x)$, for all n and all $x \in [0,1]$.*

Remark. The corresponding results for Bernstein polynomials are included in Theorem 7. We need only choose $G_x(t)$, $0 < x < 1$, to be the distribution function of the random variable taking the value 0 with probability $1 - x$ and the value $1/x$ with probability x.

3. Approximation on $[0, \infty)$ Involving Convolutions.

The applications in this subsection involve the approximation methods discussed in [*4*]. Explanation for the terminology used here can be found in that monograph.

Let $g(t)$ be a Pólya Frequency (PF) density function of a positive random variable X, and let it be so normalized that relation (47) holds for

$$G(t) = \int_0^t g(x)\, dx,$$

where b is replaced by ∞.

Let $\mathscr{S}$ denote the set of all functions $f(t)$ which are continuous on $[0, \infty)$ and grow to infinity slower than any exponential, i.e.,

$$\lim_{t\to\infty} \frac{f(t)}{e^{\eta t}} = 0, \qquad \text{for all } \eta > 0. \tag{61}$$

Let the positive linear operators $T_n(f;x)$, $n = 1, 2, \ldots$, be defined on $\mathscr{S}$ by

$$T_n(f;x) = \int_0^\infty f(tx/n)\, g^{(n)}(t)\, dt, \qquad n = 1, 2, \ldots. \tag{62}$$

The existence of these integrals for functions of $\mathscr{S}$ follows from rate of decrease properties of PF density functions (for a proof see [*4*]).

Note that since the functions 1 and t belong to $\mathscr{S}$, the remark at the end of Section II shows that Theorem 2 is applicable. Note, further, that relations (49) and (50) are satisfied by $\{T_n(f;x)\}_{n=1}^\infty$. Thus, we have

LEMMA 5. *Let $g(t)$, $\mathscr{S}$ and $T_n(f;x)$, $n = 1, 2, \ldots$, be defined as above. Then for every function $f \in \mathscr{S}$ which is convex on $[0, \infty)$, we have*

$$T_n(f;x) \geqslant f(x), \qquad x \in [0, \infty);\ n = 1, 2, \ldots. \tag{63}$$

Similar considerations yield also the following

LEMMA 6. *Let $\mathscr{S}$ be defined as above. For each $f \in \mathscr{S}$ define the Mirakyan operators $\{M_n(f;x)\}_{n=1}^\infty$, by*

$$M_n(f;x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} e^{-nx}, \qquad x \in [0, \infty);\ n = 1, 2, \ldots.$$

Then for every function $f \in \mathscr{S}$ which is convex on $[0, \infty)$, we have

$$M_n(f;x) \geqslant f(x), \qquad x \in [0, \infty);\ n = 1, 2, \ldots. \tag{64}$$

These results have been established by Karlin [*4*] by using different and more complicated techniques. The complication is compensated, however, by the fact that he derives at the same time convexity preserving properties of such operators, properties which are not derivable by the methods presented here.

4. APPROXIMATIONS ON $(-\infty, \infty)$ INVOLVING THE WEIERSTRASS KERNEL.

Let $\mathscr{W}$ denote the set of all functions $f(t)$ which are continuous on $(-\infty, \infty)$ and grow to infinity in such a way that

$$\lim_{n\to\infty} \frac{|f(t)|}{e^{\alpha t^2}} = 0, \qquad \text{for all } \alpha > 0. \tag{65}$$

Let the positive linear operators $W_n(f;x)$, $n = 1, 2, \ldots$, be defined on $\mathscr{W}$ by

$$W_n(f;x) = \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} f(t)\, e^{-n(t-x)^2}\, dt, \qquad n = 1, 2, \ldots. \tag{66}$$

These integrals clearly exist for $f \in \mathscr{W}$, and the resulting functions are defined on $(-\infty, \infty)$.

Consider the ECT-system $(1, t)$. It is obvious that 1 and t belong to $\mathscr{W}$. Simple computations yield

$$\begin{aligned} W_n(1, x) &\equiv \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-n(t-x)2}\, dt = \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-nt^2}\, dt \\ &= \left(\frac{1}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-s^2}\, ds = 1, \end{aligned} \tag{67}$$

and

$$\begin{aligned} W_n(t;x) &\equiv \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} t\, e^{-n(t-x)^2}\, dt \\ &= \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} (t-x)\, e^{-n(t-x)^2}\, dt + x \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-n(t-x)^2}\, dt \\ &= \left(\frac{n}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} s\, e^{-ns^2}\, ds + x \\ &= x. \end{aligned} \tag{68}$$

Appealing to the remark at the end of Section II, we see that Theorem 2 is applicable. Thus, using relations (67)–(68), we obtain

LEMMA 7. *Let $f(t) \in \mathscr{W}$ be convex on $(-\infty, \infty)$. Then*

$$W_n(f;x) \geqslant f(x), \qquad -\infty < x < \infty;\ n = 1, 2, \ldots,$$

where $W_n(f;x)$, $n = 1, 2, \ldots$, are defined as in (66).

We shall now prove that the "converse" result holds as well.

It is easy to demonstrate the strong centeredness of the sequence $\{W_n(f;x)\}_{n=1}^{\infty}$ on $(-\infty, \infty)$. However, this will not be sufficient under the present circumstances, since an infinite interval and unbounded functions are considered.

Retracing the proof of Theorem 4, we observe that the crucial steps are inequalities (34) and the relations following them. In order to be able to draw similar conclusions, we shall prove the following

LEMMA 8. *Let y be a point on the real line, and let*

$$d\bar{\mu}_n(t) = d\mu_n(y;t) = (n/\pi)^{1/2}\, e^{-n(t-y)^2}\, dt.$$

If $f \in \mathscr{W}$, then for each fixed pair of values $\eta > \delta > 0$, we have

$$\lim_{n\to\infty} \frac{\int_{y+\eta}^{\infty} |f(t)|\, d\bar{\mu}_n(t)}{\bar{\mu}_n[y+\delta, y+\eta]} = 0, \qquad \lim_{n\to\infty} \frac{\int_{-\infty}^{y-\eta} |f(t)|\, d\bar{\mu}_n(t)}{\bar{\mu}_n[y-\eta, y-\delta]} = 0. \tag{69}$$

Proof. We prove the right-hand limit relation. The other one can be similarly proved.

Condition (65) implies that for each $\alpha > 0$, there exists a c, such that

$$|f(t)| < c\, e^{\alpha(t-y)^2}, \qquad \text{for all } t.$$

Using this inequality, we can obtain an upper bound for the numerator in the right-hand of (69). Choosing $\alpha < 1$, we find

$$\int_{y+\eta}^{\infty} |f(t)|\, d\bar{\mu}_n(t) \leqslant c(n/\pi)^{1/2} \int_{y+\eta}^{\infty} e^{\alpha(t-y)^2} e^{-n(t-y)^2}\, dt$$

$$= c(n/\pi)^{1/2} \int_{\eta}^{\infty} e^{\alpha s^2} e^{-ns^2}\, ds.$$

Using now the fact that in the interval of integration $s > \eta$, we deduce that this expression is smaller than

$$c(n/\pi)^{1/2} \cdot 1/\eta \int_{\eta}^{\infty} s\, e^{s^2(\alpha-n)}\, ds.$$

Combining these estimates and performing the integration, we have

$$\int_{y+\eta}^{\infty} |f(t)|\, d\bar{\mu}_n(t) \leqslant \frac{c}{\eta}\left(\frac{n}{\pi}\right)^{1/2} \cdot \frac{1}{n-\alpha}\, e^{\eta^2(\alpha-n)} = \frac{c_1}{n-\alpha}\left(\frac{n}{\pi}\right)^{1/2} e^{-n\eta^2},$$

where c_1 is a constant (independent of n).

On the other hand, the following estimate for the denominator can be obtained by using the mean value theorem:

$$\int_{y+\delta}^{y+\eta} d\bar{\mu}_n(t) = \left(\frac{n}{\pi}\right)^{1/2} \int_{y+\delta}^{y+\eta} e^{-n(t-y)^2}\, dt = \left(\frac{n}{\pi}\right)^{1/2} e^{-n\xi^2},$$

where $\delta < \xi < \eta$.

Thus, for each n, the quotient on the right in (69) is smaller than

$$[c_1/(n-\alpha)]\, e^{-n(\eta^2-\xi^2)},$$

where $\xi < \eta$ and c_1 is a constant. Since this quotient clearly tends to zero as n tends to infinity, the right-hand limit relation of (69) is established.

With the aid of Lemma 8 we can prove

LEMMA 9. *Let* $f(t) \in \mathscr{W}$, *and let the operators* $W_n(f;x)$, $n = 1, 2, \ldots$, *be defined as in* (66). *If*

$$W_n(f;x) \geqslant f(x), \qquad -\infty < x < \infty; n = 1, 2, \ldots, \tag{70}$$

then $f(x)$ *is convex on* $(-\infty, \infty)$.

Proof. The proof proceeds with arguments similar to those used in the proof of Theorem 4. Repeating the construction used there, and noting that here $u_0 \equiv 1$ and $u_1 \equiv t$, we find that there exists a linear function $l(x) \equiv \alpha x + \beta + M$ which satisfies (30)–(32). We next introduce the definitions employed in (33), with the exception of M_0.

Note that $f(t) - l(t) \in \mathscr{W}$. Thus, by virtue of Lemma 8, for $\epsilon = m_0/2$, we can choose N such that for $n > N$,

$$\begin{aligned} &\int_{y+\eta_2}^{\infty} |f(t) - l(t)|\, d\bar{\mu}_n(t) < (m_0/2)\, \bar{\mu}_n[y + \delta_2, y + \eta_2], \\ &\int_{-\infty}^{y-\eta_1} |f(t) - l(t)|\, d\bar{\mu}_n(t) < (m_0/2)\, \bar{\mu}_n[y - \eta_1, y - \delta_1]. \end{aligned} \tag{71}$$

Using relations (32)–(33) and (71), we obtain the following chain of inequalities for $n > N$:

$$\begin{aligned} W_n(l - f; y) &= \int_{-\infty}^{\infty} (l - f)\, d\bar{\mu}_n \\ &\geqslant \int_{y+\delta_2}^{y+\eta_2} (l - f)\, d\bar{\mu}_n + \int_{y-\eta_2}^{y-\delta_2} (l - f) d\bar{\mu}_n \\ &\qquad - \left[\int_{-\infty}^{y-\eta_1} |l - f|\, d\bar{\mu}_n + \int_{y+\eta_2}^{\infty} |l - f|\, d\bar{\mu}_n\right] \\ &\geqslant m_0\{\bar{\mu}_n[y - \eta_1, y - \delta_1] + \bar{\mu}_n[y + \delta_2, y + \eta_2]\} \\ &\qquad - \left[\int_{-\infty}^{y-\eta_1} |l - f|\, d\bar{\mu}_n + \int_{y+\eta_2}^{\infty} |l - f|\, d\bar{\mu}_n\right] \\ &> \int_{-\infty}^{y-\eta_1} |l - f|\, d\bar{\mu}_n + \int_{y+\eta_2}^{\infty} |l - f|\, d\bar{\mu}_n \geqslant 0. \end{aligned}$$

The rest of the proof is identical with the end of the proof of Theorem 4.

We next turn to the monotonicity properties discussed in Theorem 5, and prove the analogous results here.

LEMMA 10. *Let* $f(t) \in \mathscr{W}$ *be convex on* $(-\infty, \infty)$. *Then*

$$W_n(f;x) \geqslant W_{n+1}(f;x), \qquad -\infty < x < \infty; n = 1, 2, \ldots. \tag{72}$$

Proof. This lemma can be proved by appealing (after making some simple modifications) to the previously mentioned theorem of Marshall and Proschan (see [*4*], Theorem 3.8).

We shall, however, present another simple proof, making use of the techniques of dual convexity cones. It has been shown (see [*5*]) that sufficient conditions for a measure $d\mu$ to satisfy

$$\int_{-\infty}^{\infty} f\,d\mu \geqslant 0$$

for all convex functions (under suitable growth conditions which ensure the existence of the integrals involved) are that

$$\int_{-\infty}^{\infty} d\mu = 0, \qquad \int_{-\infty}^{\infty} t\,d\mu = 0,$$

and that $d\mu$ has two sign changes, with last sign $+$. It is a matter of simple computation to convince oneself that the measure

$$\left[\left(\frac{n}{\pi}\right)^{1/2} e^{-n(t-x)^2} - \left(\frac{n+1}{\pi}\right)^{1/2} e^{-(n+1)(t-x)^2}\right] dt$$

has these properties for all $n \geqslant 1$. Thus,

$$\int_{-\infty}^{\infty} f(t)\left[\left(\frac{n}{\pi}\right)^{1/2} e^{-n(t-x)^2} - \left(\frac{n+1}{\pi}\right)^{1/2} e^{-(n+1)(t-x)^2}\right] dt \geqslant 0, \qquad n = 1, 2, \ldots,$$

for all convex functions of $\mathscr{W}$. This is equivalent to the conclusion of the lemma.

We next remark, that for all $f \in \mathscr{W}$,

$$\lim_{n\to\infty} W_n(f;x) = f(x), \qquad -\infty < x < \infty. \tag{73}$$

The proof of this statement is standard, and is therefore omitted.

Observe that an analogue of Theorem 5 can be deduced from Lemma 9 by using relation (73).

We summarize the result of this observation and Lemmas 7, 9 and 10, in the form of

THEOREM 8. *Let $f(t) \in \mathscr{W}$ and let $W_n(f;x)$, $n = 1, 2, \ldots$, be defined as in* (66). *Then*

(a) *A necessary and sufficient condition for f to be convex on $(-\infty, \infty)$ is that $W_n(f;x) \geqslant f(x)$, for all n and x.*

(b) *A necessary and sufficient condition for f to be convex on $(-\infty, \infty)$ is that $W_n(f;x) \geqslant W_{n+1}(f;x)$, for all n and x.*

Noting that a function convex on $(-\infty, \infty)$ cannot be bounded unless it is a constant, we can deduce the following

COROLLARY. *Let $f(t)$ be a bounded and continuous function on $(-\infty, \infty)$ which does not reduce to a constant. Then for each real point y, there exist sequences $\{n_i\}_1^\infty$ and $\{m_i\}_1^\infty$ such that*

$$W_{n_i}(f; y) < f(y), \qquad i = 1, 2, \ldots,$$

and

$$W_{m_i}(f; y) < W_{m_i+1}(f; y), \qquad i = 1, 2, \ldots.$$

Note added in proof. After this paper had been submitted for publication, the "converse" theorem for the special case of the Bernstein polynomials was indepently proved by L. Kosmak.

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